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A PRIMER FOR STRUCTURAL RESPONSE TO RANDOM PRESSURE FLUCTUATIONS†

by

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NOMENCLATURE

A_F, A_R	Flexible and rigid areas of cavity
a	Plate length
b	Plate width
c	Cavity sound speed
D	$Eh^3/12(1-\nu^2)$
E	Modulus of elasticity
F_m	m^{th} cavity natural mode
H_n	Plate admittance function
h	Plate thickness
I_n; I	Plate transfer function; see Eq.(28)
$K^2{}_n$	$= \dfrac{m\,\omega_n{}^2 a^4}{D}$
L_{mr}	see equation (62), et. seq.
L_x, L_y	Characteristic lengths of random pressure field
M_m	Plate generalized mass; also cavity generalized mass
m	Plate mass/area
n	normal
P; P_m	$= \dfrac{pa^4}{hD}$; see equation (57)
p	Pressure on plate
Q_m	Generalized force on plate
q_n	Generalized plate coordinate
R	Correlation function
t	Time
U_c	Pressure field convection velocity
V	Cavity volume
W_m	see equation (57)

w	Plate deflection
x, y, z	Cartesian coordinates
α, β, γ	Empirical constants for random pressure field
∇^2	Laplacian
δ^*	Boundary layer thickness
Φ	Power spectral density
ρ_m	Plate density
σ	stress
τ	Dummy time
ζ_m	Modal damping
ω_m	Modal frequency

1. INTRODUCTION

In the present report we shall treat the response of a structure to a convecting-decaying random pressure field. The treatment follows along conventional lines after Powell and others. That is, the pressure field is modelled as a random, stationary process whose correlation function (and/or power spectra) is determined from experimental measurements. Using this empirical description of the random pressure, the response of the structure is determined using standard methods from the theory of linear random processes.[2,3] The major purpose of the report is to provide a complete and detailed account of this theory which is widely used in practice (in one or another of its many variants). A second purpose is to consider systematic simplifications to the complete theory. The theory presented here is most useful for obtaining analytical results such as scaling laws or even, with enough simplifying assumptions, explicit analytical formulae for structural response. Some of these latter results are thought to be new; however, so many authors have considered various simplified versions of the general theory, the authors hesitate to claim novelty for any specific result. Hence, the title of the present report.

It should be emphasized that, if for a particular application the simplifying assumptions which lead to analytical results must be abandoned, numerical simulation of structural response time histories may be the method of choice.[4,5] Once one is committed to any substantial amount of numerical work (e.g. computer work) then the standard power spectral approach loses much of its attraction.

2. INPUT-OUTPUT RELATIONS FOR STRUCTURE

In the present section we derive the input-output relations for a flat plate. It will be clear, however, that such relations may be derived in a similar manner for any linear system.

The equation of motion for the small (linear) deformation of a uniform isotropic flat plate is

$$D\nabla^4 w + m \frac{\partial^2 w}{\partial t^2} = p \tag{1}$$

where w is the plate deflection, p the pressure loading and the other terms are defined in the Nomenclature. Associated with (1) are the natural modes and frequencies of the plate which satisfy

$$D\nabla^4 \psi_n - \omega_n{}^2 m\, \psi_n = 0 \tag{2}$$

where ω_n is the frequency and $\psi_n(x,y)$ the shape of the n^{th} natural mode. In standard texts it is shown that the ψ_n satisfy an orthogonality condition

$$\iint \psi_n \psi_m \, dxdy = 0 \qquad \text{for } m \neq n \tag{3}$$

If we expand the plate deflection in terms of the natural modes

$$w = \sum_n q_n(t)\, \psi_n(x,y) \tag{4}$$

then substituting (4) into (1), multiplying by ψ_m and integrating over the plate area we obtain

$$M_m[\ddot{q}_m + \omega_m^2 q_m] = Q_m \qquad (5)$$

$$m = 1,2,\cdots$$

where we have used (2) and (3) to simplify the result. M_m and Q_m are defined as

$$M_m \equiv \iint m \, \psi_m^2 \, dxdy$$

$$Q_m \equiv \iint p \, \psi_m \, dxdy \qquad (6)$$

$$\dot{} \equiv d/_{dt}$$

For structures other than a plate the final result would be unchanged, (5) and (6); however, the natural modes and frequencies would be obtained by the appropriate equation for the particular structure rather than (1) or (2). Hence, the subsequent development, which depends upon (5) only, is quite general.

Before proceeding further we must consider the question of (structural) damping. We shall defer a discussion of acoustic or fluid damping to a subsequent section. Restricting ourselves to structural damping only we shall include its effect in a gross way by modifying (5) to read

$$M_m \, [\ddot{q}_m + 2 \, \zeta_m \omega_m \, \dot{q}_m + \omega_m^2 q_m] \equiv Q_m \qquad (7)$$

where ζ_m is a (non-dimensional) damping coefficient usually determined experimentally. This is by no means the most general form of damping possible. However, given the uncertainty in our knowledge of damping from a fundamental theoretical viewpoint

(see [6]) it is generally sufficient to express our meager knowledge. If damping is inherent in the material properties (stress-strain law) of the structure, the theory of viscoelasticity may be useful for estimating the amount and nature of the damping. However, often the damping is dominated by friction at joints, etc., which is virtually impossible to estimate in any rational way.

Now let us turn to the principal aim of this section, the stochastic relations between loading and response. We shall obtain such results in terms of correlation functions and power spectra.

The correlation function of the plate deflection w is defined as

$$R_w(\tau;x,y) \equiv \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} w(x,y,t)w(x,y,t+\tau)dt \tag{8}$$

Using (4) we obtain

$$R_w(\tau;x,y) = \sum_m \sum_n \psi_m(x,y)\psi_n(x,y)\, R_{q_m q_n}(\tau) \tag{9}$$

where

$$R_{q_m q_n}(\tau) \equiv \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} q_m(t)q_n(t+\tau)d\tau \tag{10}$$

is defined to be the cross-correlation of the generalized coordinates, q_m. Defining power spectra

$$\Phi_w(\omega;x,y) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} R_w\,(\tau;x,y)\; e^{i\omega\tau}\, d\tau \tag{11}$$

$$\Phi_{q_m q_n}(\omega) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} R_{q_m q_n}(\tau)\; e^{i\omega\tau}\, d\tau \tag{12}$$

we may obtain from (9) via a Fourier Transform

$$\Phi_w\,(\omega;x,y) = \sum_m \sum_n \psi_m(x,y)\psi_n(x,y)\;\Phi_{q_m q_n}(\omega) \tag{13}$$

(9) and (13) relate the physical deflection w to the generalized coordinates or displacements q_m.

Consider next similar relations between physical load p and generalized force Q_m. Define the cross-correlation

$$R_{Q_m Q_n}(\tau) \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} Q_m(t)Q_n(t+\tau)\, dt \tag{14}$$

Using the definition of generalized force (6)

$$Q_m(t) \equiv \iint p(x,y,t)\;\psi_m(x,y)\; dxdy$$

$$Q_n(t+\tau) \equiv \iint p(x^*,y^*,t+\tau)\;\psi_n(x^*,y^*)\; dx^*dy^*$$

and substituting into (14) we obtain

$$R_{Q_m Q_n}(\tau) = \iiiint \psi_m(x,y)\psi_n(x^*,y^*) \bullet R_p(\tau;x,y,x^*,y^*)\, dxdydx^*dy^* \tag{15}$$

where we define the pressure correlation

$$R_p\ (\tau;x,y,x^*,y^*) \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} p(x,y,t)p(x^*,y^*,t+\tau)dt \tag{16}$$

Note that a rather extensive knowledge of the spatial distribution of the pressure is implied by (16).

Again defining power spectra

$$\Phi_{Q_m Q_n}(\omega) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} R_{Q_m Q_n}(\tau)\, e^{i\omega\tau}\, d\tau \tag{17}$$

$$\Phi_p\ (\omega;x,y,x^*,y^*) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} R_p(\tau;x,y,x^*,y^*)\, e^{i\omega\tau}\, d\tau \tag{18}$$

we may obtain from (15)

$$\Phi_{Q_m Q_n}(\omega) = \iiiint \psi_m(x,y)\psi_n(x^*,y^*) \bullet \Phi_p(\omega;x,y,x^*,y^*)\, dxdy\, dx^*,dy^* \tag{19}$$

Finally, we must relate the generalized coordinates to the

generalized forces. From (7) we may formally solve (see [2]), for example)

$$q_n(t) = \int_{-\infty}^{\infty} H_n(t-t_1)\ Q_n(t_1)\ dt_1 \tag{20}$$

where

$$H_n(t) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} I_n(\omega)\ e^{i\omega t}\ d\omega \tag{21}$$

and the "transfer function",

$$I_n(\omega) \equiv \frac{1}{M_n\ [\omega_n^2 + 2\ \zeta_n i\omega_n \omega - \omega^2]}$$

Also

$$I_n(\omega) = \int_{-\infty}^{\infty} H_n(t)\ e^{-i\omega t}\ dt$$

which is the other half of the transform pair, cf (21).

From (20) and (10)

$$R_{q_m q_n}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \iiint_{-T}^{T} H_m(t-t_1)\ H_n(t+\tau-t_2) Q_m(t_1) Q_n(t_2) \bullet dt_1\ dt_2\ dt$$

Performing a change of integration variables and noting (14),

$$R_{q_m q_n}(\tau) = \iint_{-\infty}^{\infty} H_m(\xi)\ H_n(\eta)\ R_{Q_m Q_n}(\tau-\eta+\xi)\ d\xi d\eta \tag{22}$$

Taking a Fourier Transform of (22) and using the definitions of power spectra (12) and (17), we have

$$\Phi_{q_m q_n}(\omega) = I_m(\omega)\ I_n(-\omega)\ \Phi_{Q_m Q_n}(\omega) \tag{23}$$

Summarizing, the relations for correlation functions are (9), (15), and (22) and for power spectra (13), (19) and (23). For example, substituting (19) into (23) and the result into (13) we have

$$\begin{aligned}\Phi_w(\omega;x,y) = \sum_m \sum_n \psi_m(x,y)\ \psi_n(x,y)\ I_m(\omega)\ I_n(-\omega)\\ \bullet \iiiint \psi_m\ (x,y)\ \psi_n\ (x^*,y^*)\\ \bullet\ \Phi_p\ (\omega;x,y,x^*,y^*)\ dxdydx^*dy^*\end{aligned} \tag{24}$$

This is the desired final result relating the physical loading to the physical response in stochastic terms.

Sharp Resonance or Low Damping Approximation

Often (24) is approximated further. Two approximations are particularly popular and useful. The first is the "neglect of

off-diagonal coupling". This means omitting all terms in the double sum except those for which $m = n$. The second is the "white noise" approximation which assumes that Φ_p is essentially constant relative to the rapidly varying transfer functions $I_m(\omega)$. Making both of these approximations in (24) we may obtain the mean square response

$$\overline{w^2}(x,y) \equiv R_w(\tau=0;x,y) = \int_0^\infty \Phi_w(\omega;x,y)\, d\omega$$

$$\approx \frac{\pi}{4} \sum_m \frac{\psi_m^2(x,y)}{M_m^2 \omega_m^3 \zeta_m} \iiiint \psi_m(x,y) \quad \psi_m(x^*,y^*)$$

$$\cdot \Phi_p \; (\omega_m;x,y,x^*,y^*)\, dxdydx^*dy^* \qquad (25)$$

Of course, only one or the other of these approximations may be made, rather than both. However, both stem from the same basic physical idea: The damping is small and hence, I_m has a sharp maximum near $\omega = \omega_m$. That is

$$I_m(\omega_m)\; I_n(-\omega_m) << \left|\; I_m(\omega_m) \;\right|^2$$

$$I_m(\omega_n)\; I_n(-\omega_n) << \left|\; I_n(\omega_n) \;\right|^2$$

and the "neglect of off-diagonal coupling" follows.

Also

$$\int \Phi_p \left|\; I_m(\omega) \;\right|^2 d\omega \approx \Phi_p(\omega_m) \int \left|\; I_m(\omega) \;\right|^2 d\omega$$

and (25) follows by simple integration.

Note that if we take the spatial mean square of (24) then using orthogonality (for a uniform mass distribution) one may show that the off-diagonal terms do not contribute (see Powell[1]).

Finally note that if we desire stress rather than deflection, then it may be shown that analogous to (25) one obtains

$$\overline{\sigma^2} = \frac{\pi}{4} \sum_m \frac{\sigma_m^2(x,y)}{M_m^2 \omega_m^3 \zeta_m} \iiiint \psi_m(x,y)\, \psi_m(x^*,y^*) \cdot \Phi_p\,(\omega_m;\ x,y,x^*,y^*)\, dx\,dy\,dx^*dy^* \qquad (26)$$

where σ_m is stress due to $w = \psi_m$.

3. SIMPLIFIED RESPONSE CALCULATIONS

Miles' Approximation for Spatially Well-Correlated Noise Sources
(Acoustic source characteristic length is large compared to structural dimensions)

One sometimes makes the further simplifying assumption that the acoustic pressure loading is perfectly correlated over the plate, i.e.

$$\Phi_p\ (\omega;\ x,y,x^*,y^*) = \Phi_p\ (\omega) \tag{27}$$

and the power spectra is independent of spatial coordinates. This is reasonable provided the characteristic length associated with the acoustic source is large compared to the plate length and width. For jet engine noise the characteristic length is on the order of the engine diameter and this assumption is particularly useful. For boundary layer noise, where the characteristic length is much smaller (on the order of the boundary layer thickness), it is less so. It is normally a conservative assumption in that the plate response is overestimated.

Using (27), (25) and (26) we see that

$$I \equiv \iiiint \Phi_p\ \psi_m(x,y)\ \psi_m(x^*,y^*)\ dxdy\ dx^*dy^* =$$

$$I_{Miles} \equiv \Phi_p(\omega) \quad [\iint \psi_m dxdy]^2 \tag{28}$$

Hence (25) and (26) become

$$\overline{w^2}\ (x,y) = \frac{\pi}{4} \sum_m \frac{\psi_m{}^2(x,y)}{M_m{}^2\omega_m{}^3\zeta_m} \Phi_p(\omega_m) \left[\iint \psi_m \, dxdy\right]^2 \tag{29}$$

$$\overline{\sigma^2}\ (x,y) = \frac{\pi}{4} \sum_m \frac{\sigma_m{}^2(x,y)}{M_m{}^2\omega_m{}^3\zeta_m} \Phi_p(\omega_m) \left[\iint \psi_m \, dxdy\right]^2 \tag{30}$$

These may be interpreted in a particularly helpful way by recognizing that for a <u>uniform</u>, <u>unit</u>, <u>static</u> pressure load (see (5) and (6))

$$q_m{}^s = \frac{Q_m{}^s}{M_m\omega_m{}^2} = \frac{\iint (1)\psi_m \, dxdy}{M_m\omega_m{}^2} \tag{31}$$

and thus the physical deflection due to this uniform, unit, static load is

$$w^s = \sum_m q_m{}^s \, \psi_m = \sum_m \frac{\psi_m \iint \psi_m \, dxdy}{M_m\omega_m{}^2} \tag{32}$$

and the corresponding stress is

$$\sigma^s = \sum_m q_m{}^s \, \sigma_m = \sum_m \frac{\sigma_m \iint \psi_m \, dxdy}{M_m\omega_m{}^2} \tag{33}$$

<u>If</u> the response is dominated by a single mode (both acoustically and due to the static load), then the summations in (29 - 33) may be ignored and (29) and (30) written as

$$\overline{w^2} = \frac{\pi}{4} \frac{\omega_m \Phi_p}{\zeta_m} (\omega_m) \quad (w^s)^2 \tag{34}$$

$$\overline{\sigma^2} = \frac{\pi}{4} \frac{\omega_m \Phi_p}{\zeta_m} (\omega_m) \quad (\sigma^s)^2 \tag{35}$$

The above forms are due to Miles[7].

Nearly Uncorrelated Noise Sources
(Acoustic source characteristic length is small compared to structural dimensions)

For some noise sources, e.g. turbulent boundary layers, the assumption of perfect correlation is not satisfactory and a different limiting approximation may be more useful. A typical (empirical) form of the pressure power spectra is[5]

$$\Phi_p \; (\omega; x, y, x^*, y^*) = \Phi_0(\omega) \; e^{-\frac{|x-x^*|}{L_x}} \; e^{-\frac{|y-y^*|}{L_y}} \cos \frac{\omega}{U_c} |x-x^*| \tag{36}$$

We wish to consider the case where $L_x \ll a$, $L_y \ll b$ and

L_x, L_y - (boundary layer) noise characteristic lengths in x,y directions

a, b - structural lengths in x,y directions

In Miles' approximation, it was implicitly assumed that $L_x \gg a$, $L_y \gg b$.

If we substitute (35) into (25) or (28), the result is

$$I \equiv \iiiint \Phi_p \, \psi_m(x,y) \, \psi_m(x^*,y^*) \, dxdy \, dx^*dy^* \tag{36}$$

$$\approx \quad \Phi_0 \, (\omega) \, 4 \, L_x L_y \iint \psi_m^2 \, (x,y) \, dxdy$$

To obtain the above we note that if $L_x << a$, $L_y << b$ then only for $x^* \approx x$ and $y^* \approx y$ will the integrand of right hand side of (36) contribute. Thus

$$I \approx \Phi_0 \, (\omega) \iint \psi_m^2 (x,y) \left\{ \iint e^{-\left|\frac{x-x^*}{L_x}\right|} \, e^{-\left|\frac{y-y^*}{L_y}\right|} \cos \, \omega \left|\frac{x-x^*}{U_c}\right| dx^*dy^* \right\} dxdy$$

Performing the integral over x^*, y^* (for definiteness consider a rectangular plate of dimensions, a,b)

$$I \approx \Phi_0 (\omega) L_x L_y \iint \psi_m^2 \Big\{ [2 - e^{-x/L_x} \cos kx/_{L_x} - e^{-(a-x)/L_x} \bullet \cos k \, (a-x)/_{L_x}] \bullet [2 - e^{-y/L_y} - e^{-(b-y)/L_y}] \Big\} dxdy$$

where $k \equiv \omega L_{x/U_c}$

Neglecting e^{-x/L_x}, etc. compared to 2, one finally obtains (36). Note this last step is a particularly good approximation

if ψ_m is small near $x = 0, a$ and $y = 0, b$. Conversely, if ψ_m had its maxima near the plate edges, the above might require modification (this is not the usual case).

If one still assumes the response is dominated by a single mode (a more questionable assumption for $L_x << a$ than $L_x >> a$) we see, comparing the above result for I to Miles' approximation, that the latter may be modified to give the structural response by multiplying by a factor of (square root for rms response)

$$\frac{I}{I_{Miles}} \approx \frac{4 L_x L_y \iint \psi_m^2(x,y)\, dxdy}{[\iint \psi_m dxdy]^2} \qquad (37)$$

For typical ψ_m, this ratio will be on the order of $4(L_x/a)(L_y/b)$ and hence, as expected, leads to a reduction in the estimated structural response from that given by the Miles' approximation.

Numerical Example

As an example, we have considered a clamped isotropic plate (with and without tiles) representative of those recently tested in a wind tunnel for Space Shuttle[8]. Although this was nominally a panel flutter test, some tile failures occured for reasons other than flutter. Hence, it is of interest to assess the possible impact of vibroacoustic response on these panels.

Metallic Panel Only:

The aluminum metallic panel has the following properties

$$E = 10^7 \text{ psi} \qquad a = 24''$$

$$\rho_m = .1\#/_{in^3} \qquad b = 18''$$

$$h = .05''$$

The first natural mode is approximated by

$$\psi_1 = \left[1 - \cos \frac{2\pi x}{a}\right] \left[1 - \cos \frac{2\pi y}{b}\right]$$

Using a Rayleigh-Ritz approximation the calculated natural frequency is 37 cps as compared to the measured value of 42 cps[8]. Also it is calculated that

$$w^s = .62 \text{ inches/psi}$$

and $\sigma^s = 2360 \text{ psi}/_{\text{psi}}$

From Figure 12 of Reference [9], Φ_p = .00305 psi^2/$_{cps}$ at 37 cps and it is assumed that $\zeta_1 = .01$.

Thus, using Miles' approximation,

$$\sqrt{\overline{\sigma^2}} = \sqrt{\frac{\pi}{4\zeta_1}}\ [\omega_1\ \Phi_p]^{1/2}\quad \sigma^s \tag{35}$$

$$= 7400 \text{ psi}$$

However, from Vaicaitis[5], et. al.,

$$L_x = 1.22\ \delta^* \quad , \quad L_y = .26\ \delta^*$$

where δ^* is the boundary layer thickness. Taking $\delta^* = 1$ inch and using the previously assumed ψ_1 , we compute from (37)

$$\frac{I}{I_{Miles}} = 9\ \frac{L_x}{a}\ \frac{L_y}{b} = .0066$$

which reduces the rms stress to

$$\sqrt{\overline{\sigma^2}} = 7400 \times [.0066]^{1/2} = 600 \text{ psi}$$

Addition of Tiles:

The measured value of ω_1 is 85 cps with 1" tiles resting on a soft felt strain isolator which in turn is bonded to the metallic panel[8]. From Muhlstein[9], et. al., Φ_p = .00148 at 85 cps.

It is assumed that $\zeta_1 = .02$ and

$\sigma^s_{\text{with tiles}}$ is estimated from

$$\sigma^s_{\text{with tiles}} = \frac{(\text{mass} \times \omega_1^2)_{\text{no tiles}}}{(\text{mass} \times \omega_1^2)_{\text{tiles}}} \sigma^s_{\text{no tiles}}$$

(The above could be computed more precisely from the analysis of Reference [10] if it were necessary.)

Now

$$\text{mass}_{\text{no tiles}} = .005 \ \#/_{\text{in}^2}$$

$$\text{mass}_{\text{tiles}} = (.005 + .0052) \ \#/_{\text{in}^2} =$$

metallic plate and tiles

$$\therefore \ \sigma^s = 295 \ \text{psi}/_{\text{psi}}$$

The stress in the metallic plate with tiles added is then

$$\sqrt{\overline{\sigma^2}} = 655 \text{ psi} \quad \text{(Miles' approximation)}$$

$$= 53 \text{ psi} \quad \text{corrected for } L_x, L_y \text{ effect.}$$

The bending stress in the tile itself is computed to be 65.5 psi and 5.3 psi respectively. Here, we make the conservative assumption that the isolator is ineffective and the tile bends with the metallic plate, i.e.

$$\frac{\sigma_s^{Tile}}{\sigma_s^{Aluminum}} = \frac{(Et)_{Tile}}{(Et)_{Aluminum}} = \frac{50{,}000 \times 1}{10^7 \times .05} = .1$$

The through the thickness stress (assuming tile moves up and down on isolator with no motion of metallic plate) is computed using the following values

$\sigma^s = 1\ \text{psi}/_{\text{psi}}$

$\zeta_1 = .1$

$\omega_1 = 680$ cps

$\Phi_p = .0002\ \text{psi}^2/_{\text{cps}}$

The result is

$\sqrt{\overline{\sigma}^2} = 1.2$ psi   Miles' Approximation

Correcting for L_x, L_y but now using dimensions of tile, a = b = 6", we have

$$\frac{I}{I_{Miles}} = .079$$

and $\sqrt{\overline{\sigma}^2} = 1.2 \times [.079]^{1/2} = .34$ psi.

$$\frac{\sigma_s^{Tile}}{\sigma_s^{Aluminum}} = \frac{(Et)_{Tile}}{(Et)_{Aluminum}} = \frac{50{,}000 \times 1}{10^7 \times .05} = .1$$

The through the thickness stress (assuming tile moves up and down on isolator with no motion of metallic plate) is computed using the following values

$$\sigma^s = 1\ \mathrm{psi}/_{\mathrm{psi}}$$

$$\zeta_1 = .1$$

$$\omega_1 = 680\ \mathrm{cps}$$

$$\Phi_p = .0002\ \mathrm{psi}^2/_{\mathrm{cps}}$$

The result is

$$\sqrt{\overline{\sigma^2}} = 1.2\ \mathrm{psi} \qquad \text{Miles' Approximation}$$

Correcting for L_x, L_y but now using dimensions of tile, a = b = 6", we have

$$\frac{I}{I_{Miles}} = .079$$

and $$\sqrt{\overline{\sigma^2}} = 1.2 \times [.079]^{1/2} = .34\ \mathrm{psi}.$$

Trial Modes

Sometimes it is convenient to use assumed trial modes rather than the natural modes. To determine the latter may itself involve a rather elaborate calculation.

Define

ψ_m - natural mode $\qquad \sigma_m$ - stress due to ψ_m

ϕ_j - trial mode $\qquad \sigma_j$ - stress due to ϕ_j

where, assuming for simplicity that $\Phi_p\ (\omega_m)$ is independent of position,

$$\overline{q_m^2} = \frac{\pi}{4}\ \Phi_p(\omega_m)\ \frac{[\iint \psi_m\ dxdy]^2}{M_m{}^2\omega_m{}^3\zeta_m} \qquad (38)$$

$$M_m \equiv \iint m\ \psi_m{}^2 dxdy$$

ω_m - m^{th} natural frequency

ζ_m - m^{th} damping ratio

Clarkson[11] and others advocate retaining only $m = 1$.

It will be shown (see equations (41) - (48) that

$$\iint \psi_m dxdy = b_1^m\ a\ b$$

$$M_m = mab \sum_j (b_j^m)^2 \qquad (39)$$

$$\sigma_m = \sum_j b_j^m\ \sigma_j$$

(38) and (39) may then be used to calculate $\overline{\sigma^2}$, viz.

$$\overline{\sigma^2} = \sum_m \overline{q_m^2}\, \sigma_m^2 \tag{40}$$

Here we neglect $\overline{q_m q_n}$. Call this OPTION I.

Relationships Between Natural Modes and Trial Modes:

We have

$$\psi_m = \sum_j b_j^{\,m} \phi_j \tag{41}$$

<u>where $b_j^{\,m}$ are the eigenvectors of a natural mode calculation using the trial modes, ϕ_j.</u>

We also know

$$w = \sum_m q_m \psi_m = \sum_j a_j \phi_j \;^{**} \tag{42}$$

Using orthogonality property * of ψ_m, we determine from (42) that

$$q_m = \sum_j \frac{a_j \iint \phi_j \psi_m dxdy}{\iint \psi_m^{\,2} dxdy} \tag{43}$$

Using (41) in (43) and orthogonality property * of ϕ_j

$$q_m = \frac{\sum_j a_j b_j^{\,m}}{\sum_j (b_j^{\,m})^2} \tag{44}$$

*Note we are assuming mass distribution is uniform.

**a_j are generalized coordinates associated with ϕ_j.

Thus

$$\overline{q_m q_n} = \frac{\sum_j \sum_k \overline{a_j a_k}\ b_j^{\ m} b_k^{\ n}}{[\sum_j (b_j^{\ m})^2][\sum_k (b_k^{\ n})^2]} \tag{45}$$

In practice we probably only need consider $m = n$.

We also need for some purposes to consider

$M_m \equiv \iint m\ \psi_m^{\ 2} dxdy$ and $\iint \psi_m dxdy$

Using (41)

$$\iint m\ \psi_m^{\ 2} dxdy = m \sum_{kj}\sum b_k^{\ m}\ b_j^{\ m} \iint \phi_k\ \phi_j dxdy\ ^*$$

or $$M_m = m \sum_j (b_j^{\ m})^2\ a\ b \tag{46}$$

where a,b are length, width of plate

and $$\iint \psi_m dxdy = \sum_j b_j^{\ m} \iint \phi_j dxdy$$

$$= b_1^m\ a\ b \tag{47}$$

Also from (40) and (41)

$$\sigma_m = \sum_j b_j^{\ m}\ \sigma_j \tag{48}$$

cf. (39) and (46) - (48).

* again assuming mass is uniformly distributed.

Summary:

Note this method is not limited to uniform pressure loading, even though for convenience we have made this assumption up to this point.

OPTION II

Calculate

$$\overline{a_j a_k} \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} a_j(t) a_k(t)\, dt \tag{49}$$

The a_j would have to be determined from a suitable modal dynamic analysis.

Calculate

$$\overline{\sigma^2} = \sum_j \sum_k \overline{a_j a_k}\, \sigma_j \sigma_k$$

σ_j - stress at x, y due to $w = \phi_j(x,y)$

OPTION III

Calculate $\overline{a_j a_k}$

Calculate
$$\overline{q_m q_n} = \frac{\sum_j \sum_k \overline{a_j a_k}\, b_j^{\,m}\, b_k^{\,n}}{[\sum_j (b_j^{\,m})^2][\sum_k (b_k^{\,n})^2]} \tag{50}$$

Calculate

$$\overline{\sigma^2} = \sum_m \sum_n \overline{q_m q_n}\, \sigma_m \sigma_n$$

σ_m - stress at x, y due to $w = \psi_m(x,y)$

Since ψ_m are natural modes, it should be possible to approximate

$$\overline{q_m q_n} \cong \overline{q_m^2} \, \delta_{mn}$$

Note b_j^m are eigenvectors from a trial mode eigenvalue calculation.

4. SOUND PRESSURE LEVEL IN AN ACOUSTIC CAVITY

In many instances the transmission of random pressure fluctuations from an external flow through a flexible wall into an interior cavity is of concern. Given the wall motion, the cavity pressure (sound) field may be determined. However, there also may be a "feedback" in that the cavity pressure modifies the plate motion. There also may be "feedback" due to pressure changes in the <u>external</u> flow as a result of flexible wall motion. Both give rise to acoustic or aerodynamic damping which may substantially modify the wall motion. Although this modification may be significant[4] in some instances, in this elementary discussion we ignore it.

The following analysis follows closely the derivation of Ventres[4] which is notable for its generality and simplicity. The analysis is valid for arbitrary cavity geometry and time dependent motion. The end result is an expansion for the acoustic pressure within the cavity due to motion of the walls, in terms of the normal modes of the cavity <u>with its walls assumed to be rigid</u>.

Let the cavity occupy a volume V, and be surrounded by a wall surface A, of which the portion A_F is flexible, while the remainder A_R is rigid. If the fluid within the cavity is at rest prior to motion of the wall, the fluid pressure p satisfies

the familiar wave equation, and associated boundary condition:

$$\nabla^2 p - {}^1/_{c^2} \frac{\partial^2 p}{\partial t^2} = 0 \tag{51}$$

$$\frac{\partial p}{\partial n} = -\rho \frac{\partial^2 w}{\partial t^2} \quad \text{on} \quad A_F \tag{52}$$

$$= 0 \qquad \text{on } A_R$$

In these equations ρ and c are the fluid density and acoustic velocity within the cavity, and w is the displacement of the flexible portion of the wall in the normal direction n (positive outward).

Equation (51) has normal mode solutions $F_m e^{i\omega t}$, $m = 0,1,2 \ldots$ with the following properties

$$\nabla^2 F_m = -\left(\frac{\omega_m}{c}\right)^2 F_m \tag{53}$$

$$\frac{\partial F_m}{\partial n} = 0 \quad \text{on} \quad A \tag{54}$$

$$\frac{1}{V} \int_V F_m F_n \, dV = 0 \qquad m \neq n$$

$$\equiv M_n \qquad m = n \tag{55}$$

Note that equation (53) has the solution $\omega_0 = 0$, $F_0 = 1$. All other frequencies ω_m , $m = 1,2, \ldots$ are positive, however.

The wave equation (51) can be transformed into a set of ordinary differential equations in time by using Green's Theorem in the form:

$$\int_V (p\nabla^2 F_m - F_m \nabla^2 p)\, dV = \int_A \left(p \frac{\partial F_m}{\partial n} - F_m \frac{\partial p}{\partial n}\right) dA \tag{56}$$

By defining

$$P_m \equiv \frac{1}{\rho c^2 V} \int_V pF_m \, dV$$
$$W_m \equiv \frac{1}{V} \int_{A_F} w\, F_m \, dA \tag{57}$$

and making use of the fact that p and F_m satisfy equations (51) and (53), and boundary conditions (52) and (54), the following ordinary differential equations are obtained from (56):

$$\ddot{P}_m + \omega^2_m \; P_m = - \ddot{W}_m \tag{58}$$

A dot ($\dot{}$) denotes differentiation with respect to time. The quantities P_m and W_m are the coefficients in normal mode expansions for the pressure and plate deflection, e.g.

$$\frac{p}{\rho c^2} \equiv P = \sum_m \frac{P_m}{M_m} \; F_m \tag{59}$$

Since the normal modes F_m satisfy the homogeneous boundary condition (54) on the entire wall surface A, the normal derivative of expression (59) does not converge uniformly on the flexible portion, A_F, of the wall surface. Expression (59) is suitable, however, for calculating the pressure itself throughout the cavity and everywhere on the wall surface, including the flexible portion.

Equations (58) may be easily solved for arbitrary wall motions. The solution appropriate for the usual initial conditions, $p = \frac{\partial p}{\partial t} = 0$, at $t = 0$ is

$$P_0 = - W_0(t)$$
$$P_m(t) = - \int_0^t \dot{W}_m(\tau) \cos \omega_m(t-\tau)\, d\tau \tag{60}$$

In keeping with the intention that the cavity be in equilibrium at $t = 0$ and for all previous time, it has been specified that $w = 0$ and $\frac{\partial w}{\partial t} = 0$ at $t = 0$ in (60).

The wall deflection w is often expressed as a series of the form

$$w = \sum_m q_m \psi_m \tag{61}$$

in which the modal functions Ψ_m are defined over the region A_F, their properties being determined by structural considerations.

In this situation the quantities W_m are, from (57)

$$W_m = \frac{A_F}{V} L_{mr} q_r \tag{62}$$

where

$$L_{mr} \equiv \frac{1}{A_F} \int_{A_F} F_m \psi_r \, dA$$

Equations (58) become

$$\ddot{P}_m + \omega^2{}_m P_m = - \frac{A_F}{V} L_{mr} \ddot{q}_r \tag{63}$$

In the present context the relationships between structural motion and cavity pressure power spectra are of interest.

Taking a Fourier Transform of (63), we have

$$(-\omega^2 + \omega_m{}^2) \, \overline{P}_m = \frac{A_F}{V} L_{mr} \omega^2 \, \overline{q}_r \tag{64}$$

Defining power spectra in the usual way, we obtain

$$\Phi_{P_m P_n} = |H_{mnrs}|^2 \, \Phi_{q_r q_s} \tag{65}$$

where $H_{mnrs} \equiv H_{mr} \, H_{ns}$ (66)

and $$H_{mr} \equiv \frac{A_F}{V} \frac{L_{mr} \omega^2}{(-\omega^2 + \omega_m{}^2)} \tag{67}$$

Alternatively using (60) and (62) one can relate the correlation

functions of P_m and q_r. Subsequently, one can relate q_r to w via (61) and P_m to P via (59). Eventually, we may obtain a relationship between the power spectra of P and w, the desired result. Since much of this analysis parallels that of earlier sections on the relationship of external pressure to structural motion we omit details here.

5. EMPIRICAL EXPRESSION FOR RANDOM PRESSURE SPECTRA

From (25) it is seen that the pressure power spectra is required. This quantity is usually determined by measurement with an empirical curve fit made to the data. Here we shall concentrate on boundary layer pressure fluctuations although the results will be qualitatively representative of other random fluid motion such as engine noise, etc.

Let us begin by considering the correlation function (16). Various authors have given empirical equations. Typical is the one given by Y. L. Lin[3]

$$R_p\,(\tau;\ x,y,x^*,y^*) = \overline{p^2}\ e^{-\alpha|x^*-x|}$$

$$\cdot\ e^{-\beta\,|(x^*-x)-U_c\tau|_e\ -\gamma|y^*-y|} \qquad (68)$$

$\overline{p^2}$, U_c, α, β, γ

are constants chosen to fit the experimental data. Note that R_p depends only on the <u>difference</u> of the spatial coordinates. To the extent that this is an accurate approximation this greatly reduces the amount of experimental information required.

Let us consider for a moment physical interpretations of the constants. From (68)

$$R_p\ (\tau;x,y,x^*,y^*) = R_p(\tau;x^*-x,y^*-y) \tag{69}$$

For $\quad x-x^* = y-y^* = 0,$

$$R_p\ (\tau;0,0) = \overline{p}^2 \tag{70}$$

Hence, $\overline{p}^2$ is is the mean square pressure, see (16), at any point on the plate. The mean square at all points is the same by virtue of the assumption that only the spatial differences appear in (68).

Next consider $|x^*-x| \neq 0$. The second exponential factor of (68) represents a convecting pressure form traveling at velocity U_c. For fixed x^*-x and y^*-y, R_p will have a maximum at $\tau_{max} = (x^*-x)/_{U_c}$.

Finally, the constants α, β, γ may be written

$$\alpha \equiv c_1/_{\delta^*}$$

$$\beta \equiv c_2/_{\delta^*}$$

$$\gamma \equiv c_3/_{\delta^*}$$

where δ^* is the boundary layer thickness and c_1, c_2, c_3 are non-dimensional constants on the order of .01-1. Of course, it is expected on physical grounds that the spatial scale would be roughly of the size of δ^*.

Using the definition of power spectra (18) and the empirical expression (68) we have

$$\Phi_p\ (\omega;x^*-x,y^*-y) \equiv \frac{1}{\pi}\int_{-\infty}^{\infty} R_p\ (\tau;x^*-x,y^*-y)e^{i\omega\tau}\ d\tau \tag{72}$$

$$= \frac{\overline{p}^2}{\pi}\left[\frac{2\beta U_c}{(\beta U_c)^2+\omega^2}\right] e^{-\alpha|x^*-x|-\gamma|y^*-y|-i\omega\frac{(x^*-x)}{U_c}}$$

This equation is the input needed for (24) or (25) to carry our analysis further. Alternatively Φ_p may be available directly from experimental data.

It should be noted that more complicated correlation and power spectra functions are sometimes used. See (36) and Reference [5].

6. CONCLUDING REMARKS

A review of power spectral methods for determining linear response of structures to random pressure fluctuations has been given. Various simplifying assumptions are made for the purpose of obtaining useful explicit formulae for structural response. The transmission of sound through a flexible structure into an interior cavity is also briefly treated.

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